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Abstract

Rudolph E. Kalman is mainly known for the Kalman filter, first published in 1960. In this year, he published two equally important contributions, one about linear state space system theory and the other about linear quadratic optimal control theory. These three domains are intertwined in the later theory of linear quadratic Gaussian control. An extended version of linear quadratic optimal control is put into practice in an example of cooperation in population ecology.

Keywords: Kalman filter, linear quadratic Gaussian control, linear system, state space, cooperation

1 Introduction

Rudolf Emil Kalman passed away on the 2nd of July 2016. His name will remain tied to the *Kalman filter*. The purpose of the Kalman filter is to estimate time-varying variables from noisy and incomplete measurements, using a (noisy) model of the underlying dynamics. Kalman objected that his filter belonged to "estimation theory," arguing that this term had come to name a body of non-science. He later also denied that the filter was based on probabilities; as much as he refused to reduce it to a mere algorithm, because, for him, the filter was to be acknowledged as a theory on its own.

Because the filter allows one to estimate a variable which is not directly measured, it has sometimes been publicized by computer scientists as a "software sensor". The filter and its extensions are now used in many domains, be they industrial and technological, or in the health, social, biological, or earth sciences. It is also implemented in global positioning systems (GPS).

In the same year 1960 when Kalman published his article about the filter (Kalman, 1960b), he also published a major article on linear quadratic optimal control (Kalman, 1960a), which he had earlier presented at a conference in 1959, and an also major article on system theory (Kalman, 1960c), completed by Kalman (1962, 1963). These contributions were ground-breaking for control theoreticians. They transformed control theory and linear system theory, and their influence went beyond the mere domain of filtering and prediction. Kalman, in this year 1960, also published important articles on the stability of linear dynamical systems (Kalman and Bertram, 1960a, b).

Bolstered by the advent of the digital computer, the theories presented in (Kalman, 1960a, b) were extensively put into use in the "Automatic Synthesis Program" involved in the Apollo lunar landing program, which started also in 1960. A new theory has seldom been adopted so quickly by practitioners and for such significant a project. This had been the case, though, of the Wiener filter, which was conceived in 1940 for the anti-aircraft radar, and had been embargoed until 1949 for its military sensitivity.

We shall present here Kalman's contributions in the early 1960's, with a focus on control theory and with a toy application in population ecology.

2 System theory

2.1 A paradigmatic change

2.1.1 State of the theory before 1960

Before 1960, system and control theories were confined to linear time-invariant (LTI) systems, mostly so-called "monovariable" ones, which are characterized by scalar signals.

A (linear) dynamical system is a device which is excited by a time-varying signal, called the *input*, to produce a time-varying signal, called the *ouput*. For mathematicians, a linear system amounts to a linear operator, transforming a time function into another one. Because this excitation happens in real time, the transformation must satisfy the principle of *causality* (the current output does not depend on future inputs). Moreover, description and analysis based on the theory available at that time require linearity, time-invariance, and some other properties about behavior at infinity.

Mathematicians had developed an elegant and powerful way for handling such transformations, through the use of the (quite esoteric) *Laplace transform*, which leads to the representation of a system through a "transfer function," which is a ratio of polynomials of a complex variable. An important point is that the transfer function is easily obtained from a linear differential equation governing the system.

The transfer function has the property of transforming a cascade of systems (whereby the output of a system is the input of the next one) into simple products of their transfer functions. This allows the analysis of *feedback* systems, where the output is re-introduced as a component of the input of the same system. This loop is necessary to servomechanisms. The concept of transfer function also led to the Wiener filter, which, at that epoch, was the standard tool in signal processing.

The transfer function and the Wiener filter were intimately tied to the *frequency response* of the system, that is, its behavior if excited by sinusoidal signals of various frequencies.

A clever trick of Wiener filtering was to consider a noisy signal to be "filtered" (net its noise) as the output of a linear system excited by a noise with adequate statistical properties. Kalman used this move in the Kalman filter, advising one of the authors (Pierre Bernhard): "Take the Kalman filter, which, as everybody knows, was invented by Wiener".

2.1.2 Innovations in the year 1960

Kalman represented the transformation of inputs into outputs by the mediation of an internal *state* of the system, consisting of a vector of real variables, whose temporal variations are governed by a first-order differential equation in continuous time or by a first-order difference equation in discrete time. This is why this description of the system is called *internal*, the classical one then being called *external*. In that representation, the input acts on the dynamics of the state, and the state instantaneously produces an output. Because the state is a vector and all relations are linear, linear algebra is appropriate. This mathematical machinery allows the consideration of vector-valued inputs and outputs. Moreover, if the matrices defining the system vary over time, the system is no longer time invariant. Part of realization theory, which lies at the heart of linear system theory, not only concerns linear time-invariant systems, but also can be extended to systems that are not invariant over time, including filtering and optimal control.

The concept of the state of a system was imprecisely known as "*a set of numbers from which the entire future behavior of the plant may be determined provided that the future inputs of the plant are known*" (Kalman and Koepcke, 1958: 1821); for example positions and velocities in a mechanical system, intensities in inductors and charges of capacitors in an electrical circuit. Kalman (1960c) mentioned that it is the *smallest* such set. This property was present in the (earlier) concept of Nerode equivalence of formal languages and used in the theory of automata (Kalman et al., 1969). The intimate link of the concept of state with first-order differential equations was now acknowledged, but often confined to the derivation of the transfer function of the linear time-invariant system. The direct use of differential (or difference) equations in optimization had become more common in the late fifties under the influence of Bellman's Dynamic Programming (Bellman, 1957). Kalman's bold move was to implement this use in the very definition of a linear system and to investigate its properties.

The internal description avoids using the Laplace transform, with the consequence that systems no longer need to be time invariant. Signals now amount to time functions, which justifies the term *time domain*, in contrast to the term *frequency domain*. The tools developed by Kalman and after him are treated with the calculus of variations for optimal control, and with Markov processes for filtering.

The advent of the digital computer and of direct digital control led Kalman and others to develop a discrete-time theory, originally as a theory of *sampled data systems*, which consists of considering continuous-time systems only at discrete instants. The parallel was natural and elegant in the new theory. Kolmogorov (1941), independently of Wiener, had earlier pioneered a discrete-time version of the Wiener filter.

The transfer function of a time-invariant system in internal form results from a simple algebraic formula. Conversely, finding the internal representation of a system given in external form is a deep question, which involves a throrough analysis of a linear system in internal form. This is the object of realization theory.

2.2 Realization theory

Kahlman's contribution to system theory began with his article "On the general theory of control systems" (Kalman, 1960c), but the founding publications, which we shall follow here, are Kalman (1962, 1963).

Definition 1 *A* realization *of a linear (or affine) input-output transformation is a representation in internal form as expressed in Eq.* (1) *and* (2)*, or* (3) *and* (4) *below.*

Let $x \in \mathbb{R}^n$ be the state of the system (n is called the dimension of the realization), $u \in \mathbb{R}^m$ the input, or control, and $y \in \mathbb{R}^p$ the output. The interesting cases are $m \leq n$ and $p \leq n$. We use Newton's notation for time derivatives: $\dot{x} = \mathrm{d}x/\mathrm{d}t$. A continuous-time system is of the form

$$\dot{x}(t) = Fx(t) + Gu(t)\,, \tag{1}$$

$$y(t) = Hx(t)\,, \tag{2}$$

and in discrete time

$$x(t+1) = Fx(t) + Gu(t)\,, \tag{3}$$

$$y(t) = Hx(t)\,. \tag{4}$$

Kalman argued against the possibility of augmenting the output equation with a term $+Ju(t)$.

For the sake of completeness, we mention that if all three matrices F, G, and H are constant, then the system is linear time-invariant, and its transfer function is

$$\mathcal{H}(s) = H(sI - F)^{-1}G. \tag{5}$$

Cramer's rule (Birkhoff and Mac Lane, 1967) implies that $\mathcal{H}(s)$ is a matrix of (strictly proper) rational fractions of s, with the characteristic polynomial of F as their common denominator, and thus the poles of $\mathcal{H}(s)$ are the eigenvalues of F.

Even without referring to the transfer function, the transformation from input to output induced by these equations is unchanged when changing the basis in the state space, or equivalently if we use the state $\xi = Tx$, where T is an invertible matrix. The continuous-time system becomes

$$\dot{\xi} = TFT^{-1}\xi + TGu, \tag{6}$$

$$y = HT^{-1}\xi. \tag{7}$$

Hence changing (H, F, G) into (HT^{-1}, TFT^{-1}, TG) leaves the system unchanged, with the same transfer function if the system is time-invariant. This points out that the same input-output system may correspond to a new representation, which is then not unique. The lack of uniqueness also occurs with a higher dimensional vector, say z, made of x and a vector ξ of arbitrary dimension:

$$\begin{aligned} z &= \begin{pmatrix} x \\ \xi \end{pmatrix}, \\ \dot{z} &= \begin{pmatrix} F & 0 \\ A & B \end{pmatrix} z + \begin{pmatrix} G \\ C \end{pmatrix} u, \\ y &= (H \quad 0) z . \end{aligned} \tag{8}$$

A, B, and C are arbitrary matrices. They play no role, because ξ does not influence y, neither directly nor through x. Therefore, realizations of different dimensions may represent the same input-output system. In the case of Eq. (8), it is trivial but if a change of the basis such as the previous one mixes x and ξ, then the dimension in excess may be more difficult to detect. Moreover, other cases are possible.

The solution of this problem involves *controllability* and *observability.* A state is controllable if there exists a control function $u(\cdot)$ that drives the system from this state to the origin. The system is said to be *completely controllable* if every state is controllable.

For the theory to encompass both continuous- and discrete-time systems, we now use the later concept of *reachability.* A state is *reachable* if there exists a control function that drives the system from the origin to that state. A system is *completely reachable* if every state is reachable. The two concepts are equivalent for continuous-time systems, but not for discrete-time ones, unless the matrices $F(t)$ are invertible for all t. Consequently, a system is completely reachable if the application $u([t_0, t_1]) \mapsto x(t_1)$, which is linear if $x(t_0) = 0$, is onto (that is, surjective) for some $t_1 > t_0$.

A state $x_0 \neq 0$ is *unobservable* if the output of the "free" system, corresponding to $u(\cdot) = 0$, initialized at x_0, is $y([t_0, t_1]) = 0$ for any $t_1 \geq t_0$. A system is *completely observable* if no state is unobservable. Consequently, the system is completely observable if the application $x(t_0) \mapsto y([t_0, t_1])$, which is linear if $u(\cdot) = 0$, is one to one (that is, injective) for some $t_1 > t_0$.

Kalman (1960a) also gave efficient criteria to examine these properties. In the case of time-invariant systems, the Kalman criteria are expressed in terms of the ranks of composite matrices:

Theorem 1

$$(F\ G) \textit{ completely reachable} \Leftrightarrow \operatorname{rank}(G\ FG\ F^2G \ldots F^{n-1}G) = n\,, \tag{9}$$

$$(H\ F) \textit{ completely observable} \Leftrightarrow \operatorname{rank}\begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{pmatrix} = n \tag{10}$$

Kalman (1960a) also gave simple criteria for systems that are not time-invariant. They are less algebraic, more analytic, but they share the property called *duality* (we use prime for transposed), defined as:

$$(F\ G) \text{ completely reachable} \Leftrightarrow (G'\ F') \text{ completely observable} \tag{11}$$

$$(H\ F) \text{ completely observable} \Leftrightarrow (F'\ H') \text{ completely reachable.} \tag{12}$$

We shall see that this duality concerns optimal control and filtering. Kalman (1960c) analyzed the duality between the Wiener filter and the linear quadratic regulator. He pointed out that it is also related to the known duality between the differential equations $\dot{x} = Fx$ and $\dot{p} = -F'p$ or the difference equations $x(t+1) = Fx(t)$ and $p(t) = F'p(t+1)$, which leave the inner product $p'x$ invariant.

But the whole extent of duality in the linear quadratic Gaussian theory, which we present below, remains difficult to explain; the more so that it mysteriously extends into modern $\mathcal{H}_\infty$-optimal control (Başar and Bernhard, 1995).

We use these concepts in realization theory with the formal definition:

Definition 2 *A completely reachable and completely observable realization is called* canonical.

The main theorem is:

Theorem 2 *A realization is minimal (has a state space of minimum dimension) if and only if it is canonical. It is unique up to a change of the basis in the state space.*

Kalman (1962) further showed that the state space of any linear system in internal form, not necessarily time-invariant (that is, a system such as Eq. (1) and (2) with matrices H, F, and G depending continuously on time) can be decomposed canonically as the direct sum of four subspaces —which can vary over time if the system is not time-invariant— as represented in the left-hand side diagram of Figure 1, or more classically, in block diagram, as in Kalman (1963), and reproduced in all textbooks since then. A is the subspace of reachable but unobservable states; B the subspace of reachable and observable states; C the subspace of unreachable and unobservable states; D the subspace of unreachable but observable states. Using a basis adapted to

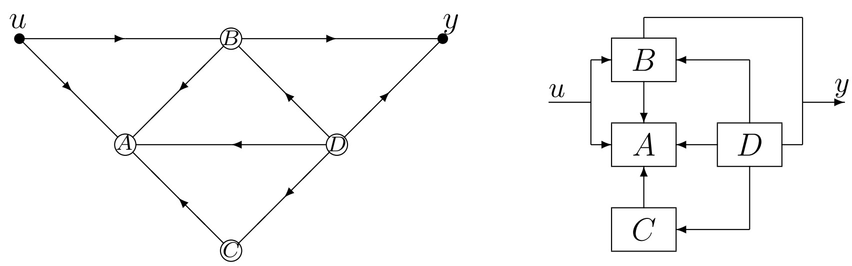

Figure 1: The canonical decomposition of a linear system in internal form. Source: Kalman (1962).

that decomposition yields a canonical decomposition of the system matrices, which takes the form:

$$F = \begin{pmatrix} F_{AA} & F_{AB} & F_{AC} & F_{AD} \\ 0 & F_{BB} & 0 & F_{BD} \\ 0 & 0 & F_{CC} & F_{CD} \\ 0 & 0 & 0 & F_{DD} \end{pmatrix}, \quad G = \begin{pmatrix} G_A \\ G_B \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

$$H = \begin{pmatrix} 0 & H_B & 0 & H_D \end{pmatrix}. \tag{14}$$

The subspaces B, C, and D are not uniquely defined, contrary to the decomposition of the matrices of the system (H, F, G), up to changes of the basis within each of the four subspaces.

2.3 Compensator design by pole placement

The Kalman filter was the first *observer*, also called "observing system" in Kalman (1960c), who proposed either an "optimal observer" or an "optimal observing system" in terms of the total number of time steps necessary to retrieve the state in a discrete-time system. The term "observer" was coined by Luenberger (1964), who extended the concept in a less explicit and more complicated form, ignoring the stability argument we use below in paragraph 4.1.2. The equations of the Kalman filter are, in discrete time as it appeared originally in Kalman (1960b):

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + K(y(t) - H\hat{x}(t)), \tag{15}$$

or in continuous time (Kalman and Bucy, 1961):

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + K(y(t) - H\hat{x}(t)), \tag{16}$$

providing an estimate $\hat{x}(t)$ of the state, optimal in the sense that it minimizes the expected squared L^2 norm of the *error signal* $\tilde{x}(t) = x(t) - \hat{x}(t)$. The natural idea, proposed by Kalman (1960c) for monovariable discrete-time systems, is to associate such an observer with a control law

$$u(t) = -C\hat{x}(t). \tag{17}$$

We now examine the choice of the gains K and C.

With a change of variables from $(x, \hat{x})$ to $(x, \tilde{x})$, one can prove the *principle of separation of the dynamics*:

Theorem 3 (Separation of dynamics) *The set of the eigenvalues of the dynamic matrix of the closed loop observer-controller is the union of the eigenvalues of the "controller" $F-GC$ and those of the "observer" $F-KH$.*

Wonham (1967) extended the *pole shifting theorem* from single-input systems (Luenberger, 1964) to multi-input ones (early alternative proofs are in Heymann (1968) and Davison (1968)):

Theorem 4 *If the pair (F, G) is completely reachable, then, given any monic polynomial $p(z)$ of degree n, there exists a matrix C such that the characteristic polynomial of $F - GC$ is p. Dually, if the pair (H, F) is completely observable, the characteristic polynomial of $F - KH$ can be assigned to any desired monic polynomial of degree n by the choice of K.*

We deduce an argument of pure system theory in favor of the proposed control structure, and a means of choosing C and K (see paragraph 4.1.2 below).

Moreover, in the discrete-time case, consequently also in the sampled-data problem of any digital control, the observer behaves as a one-step predictor, because the system estimates $\hat{x}(t+1)$ for $x(t+1)$ with the data $y(\tau)$, $\tau \leq t$. The control algorithm then has one time step at its disposal to compute the control $u(t+1) = -C\hat{x}(t+1)$, with the knowledge of the gain C, which has been computed off line.

3 Linear quadratic Gaussian (LQG) theory

3.1 Linear quadratic (LQ) optimal control

The topic covered here is partially addressed in Kalman (1960c), but the authoritative article is Kalman (1960a), whose notation we adopt for the most part.

In his introduction, Kalman (1960a: 102) stated: "This problem dates back, in its modern form, to Wiener and Hall at about 1943." He also cited Newton Jr et al. (1957), as representing the state of the art.

Therefore, in its infinite horizon (optimal regulator) form, Kalman's linear quadratic optimal control theory was not new. However, the solution provided by the theory then available (Newton Jr et al., 1957) was in terms of spectral factorization, which is comparable to the Wiener Filter and did not easily lead to efficient algorithms, particularly so for "multivariable" problems. It could not be extended to finite-horizon problems either. Actually, one finite-horizon non-homogeneous scalar-control linear-quadratic optimization problem is solved in Merriam III (1959), with the correct Riccati equation and the linear equations for the non-homogeneous terms, although the latter are difficult to recognize.

3.1.1 Finite-horizon problem

The theory of linear-quadratic optimal control began with the investigation of a finite-horizon optimal control problem, which is then not time-invariant. This problem involves quadratic forms. We express it as: for any positive definite or semi-definite $\ell \times \ell$ matrix M and z a ℓ-vector,

$$\langle z, Mz\rangle = z'Mz = \|z\|_M^2 \,. \tag{18}$$

The problem is (presentation in a more general form in Kalman (1960a)):

Linear quadratic optimal control problem Given the system of Eq. (1) and (2), with all system matrices (possibly piecewise) continuous functions of time, and $x(t_0) = x_0$, and given the symmetric $n \times n$ matrix $A \geq 0$, and the symmetric (piecewise) continuous $n \times n$ matrix function $Q(t) \geq 0$ and $m \times m$ matrix function $R(t) > 0$, find the control law, if it exists, that minimizes the performance index

$$V(x_0, t_0, t_1; u(\cdot)) = \|x(t_1)\|_A^2 + \int_{t_0}^{t_1} \left(\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2\right) \mathrm{d}t\,. \tag{19}$$

While Kalman and Koepcke (1958) and Kalman (1960c) explicitly use Bellman (1957) and discrete dynamic programming, which was a strong incentive to use the state-space representation, Kalman (1960a) for the continuous-time problem used the theory of Carathéodory (1935). Conversely, Bellman (1957) did not refer to Carathéodory. Yet, continuous-time dynamic programming is mostly a re-discovery of Carathéodory's theory, with Bellman's *return function* playing the same role as Carathéodory's *principal function.*

Define a symmetric matrix function $P(t)$ as the solution, if it exists, of the matrix Riccati equation (where all matrices are time-dependent)

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH\,, \qquad P(t_1) = A. \tag{20}$$

(By Cauchy's theorem, there exists a solution on some open time interval (t_2, t_1). However, there is no guarantee a priori that a solution exists over the time interval $[t_0, t_1]$, because the solution might diverge to infinity before reaching t_0 by decreasing values.)

The full theorem is:

Theorem 5

1. *The Riccati Eq.* (20) *has a solution* $P(t) \geq 0$ *over* $[t_0, t_1]$ *for every* $t_0 < t_1$.

2. *The solution of the linear quadratic optimal control problem is given in state feedback form by*

$$u(t) = -C(t)x(t)\,, \qquad C(t) = R(t)^{-1}G'(t)P(t)\,, \tag{21}$$

3. *and the optimal value of the performance index is*

$$V^0(x_0, t_0, t_1) = \|x_0\|^2_{P(t_0)}\,. \tag{22}$$

We emphasize that the Riccati Eq. (20) and the optimal feedback gain in Eq. (21) are the duals of the Riccati equation and gain associated with the Kalman filter (see section 3.2)

3.1.2 Optimal regulator (infinite-horizon) problem

For the sake of simplicity, we give here only the linear time-invariant version, which is the only one that the previous theory could handle. Kalman (1960a) however gave also the solution for a non time-invariant problem.

Optimal regulator problem Given the time-invariant linear system Eq. (1) and (2) with initial state $x(0) = x_0$, and a non-negative definite $p \times p$ matrix

Q and a positive definite $m \times m$ matrix R, find the control, if it exists, that minimizes the performance index

$$V(x_0; u(\cdot)) = \int_0^\infty \big(\|y(t)\|_Q^2 + \|u(t)\|_R^2\big) \mathrm{d}t\,. \tag{23}$$

This study requires the introduction of both controllability and observability. Indeed, in his introduction, Kalman states that "the principal contribution of the paper lies in the introduction and exploitation of the concepts of *controllability* and *observability*."

Retrospectively, he could also have quoted the Riccati Eq. (20). We use here its algebraic version, where all matrices are now constant:

$$PF + F'P - PGR^{-1}G'P + H'QH = 0\,. \tag{24}$$

The full theorem is:

Theorem 6

1. *If the pair (F, G) is completely controllable, then*

   (a) *the solution $P(t)$ of the Riccati Eq.* (20) *has a limit $\bar{P}$ as $t \to -\infty$, which solves the algebraic Riccati Eq.* (24)*,*

   (b) *the solution of the optimal regulator problem in state feedback form is*

$$u(t) = -Cx(t)\,, \qquad C = R^{-1}G'\bar{P}\,, \tag{25}$$

(c) and the optimal value of the performance index is

$$V^0(x_0) = \|x_0\|^2_{\bar{P}} \tag{26}$$

(it is actually sufficient that the pair (F, G) *is* stabilizable*, that is,* $\exists D : F - GD$ *stable);*

2. *If furthermore the pair* (H, F) *is completely observable and* $Q > 0$*,* $\bar{P}$ *is positive definite and the system governed by Eq.* (25) *is asymptotically stable (for the stability result, it is sufficient that* $Q \geq 0$ *and* $(Q^{1/2}H, F)$ *is* detectable*, that is,* $\exists L : F - LQ^{1/2}H$ *stable).*

The duality we pointed out in the finite-horizon problem holds here, making the optimal regulator dual to the stationary Kalman filter, that is, to a realization of the Wiener filter.

3.1.3 Discrete-time case

Kalman (1960b, c), building on Kalman and Koepcke (1958: Appendix), which dealt with sampled data control (which, as we mentioned before, consists of considering continuous-time systems only at discrete instants), of a continuous-time system provides the equivalent discrete-time results. The system is made of Eq. (3) and (4), and the performance index is:

$$V(x_0, t_0, t_1; \{u(\cdot)\}) = \|x(t_1)\|_A^2 + \sum_{t=t_0}^{t_1-1} \left(\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2\right). \tag{27}$$

The Riccati differential equation is replaced by the so-called discrete Riccati equation (the system matrices may all depend on time) or its "algebraic" version (with all system matrices constant), where $P(t) = P(t+1) = \bar{P}$:

$$\begin{aligned} &P(t) = F'P(t{+}1)F - F'P(t{+}1)G(G'P(t{+}1)G + R)^{-1}G'P(t{+}1)F + H'QH, \\ &P(t_1) = A\,. \end{aligned} \tag{28}$$

The optimal feedback control is

$$u(t) = -C(t)x(t) \quad \text{and} \quad C(t) = (G'P(t+1)G + R)^{-1}G'P(t+1)F. \tag{29}$$

Both the finite- and the infinite-horizon results follow exactly as for the continuous-time case, with the same controllability and observability conditions.

3.2 The Kalman Filter

For the sake of completeness, and to stress duality, we briefly review the Kalman filter (Kalman, 1960b; Kalman and Bucy, 1961). Existence and stability properties for both the finite and the infinite horizon cases derive directly from those of the dual linear-quadratic control problem.

3.2.1 Discrete time

We start with the discrete-time problem, after Kalman (1960b) (where there is neither added noise in the measurement equation nor control). We consider a discrete-time linear system excited by *white noise* and a known control $u(\cdot)$:

$$x(t+1) = F(t)x(t) + G(t)u(t) + D(t)v(t)\,, \quad x(t_0) = x_0\,, \tag{30}$$

$$y(t) = H(t)x(t) + w(t), \tag{31}$$

where $(v(t), w(t))$ is a Gaussian random variable with zero mean and known covariance, independent of all the $(v(\tau), w(\tau))$ for $\tau \neq t$. In the simplest case, $v(t)$ is also independent of $w(t)$, but this is not necessary to the theory. A possible non-zero cross-correlation between $v(t)$ and $w(t)$ is dual to the presence of a cross term $x'Su$ in the quadratic performance index of linear-quadratic control. The noise is characterized by its covariance matrix (with $\delta_{t,\tau}$ the Kronecker symbol):

$$\mathbb{E}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \begin{pmatrix} v'(\tau) & w'(\tau) \end{pmatrix} = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix} \delta_{t,\tau}\,. \tag{32}$$

The initial state is also a Gaussian random variable of known distribution:

$$\mathbb{E}(x(t_0)) = \hat{x}_0\,, \qquad \mathbb{E}(x(t_0) - \hat{x}_0)(x(t_0) - \hat{x}_0)' = \Sigma_0. \tag{33}$$

The problem is to compute the conditional mathematical expectation:

$$\hat{x}(t) = \mathbb{E}(x(t)|y(\tau), \tau < t)\,. \tag{34}$$

The solution is of the form of Eq. (15) initialized at $\hat{x}(t_0) = \hat{x}_0$, where the gain K is given through the error covariance matrix $\Sigma(t) = \mathbb{E}(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))'$, which is solution of the discrete Riccati Eq. (35), which is dual of Eq. (28), and by the formula in Eq. (36), which is dual of Eq. (29):

$$\Sigma(t+1) = F\Sigma(t)F' - F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}H\Sigma(t)F' + DVD'\,, \tag{35}$$

$$K(t) = F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}\,. \tag{36}$$

The time-invariant infinite-horizon case is the internal form of the Kolmogorov filter.

Continuous-time Given a continuous-time system in internal form excited by "white noises" (quotes because such a thing does not exist rigorously; it is an "engineering" presentation) involved both in the dynamics and in measurement, as in the previous subsection on discrete time, but in continuous time, with

$$\mathbb{E}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} (v'(\tau)\ w'(\tau)) = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix} \delta(t-\tau) \tag{37}$$

(we use the Dirac δ as in Kalman and Bucy (1961), but probabilists nowadays write the system differently, with a different ontology, in terms of diffusions, Itô calculus, and without "white noise"), the conditional expectation in Eq. (34) we are looking for is the solution of the continuous-time observer of Eq. (16) with the dual formulas from linear-quadratic control:

$$\dot{\Sigma} = F\Sigma + \Sigma F' - \Sigma H' W^{-1} H \Sigma + DVD' \,, \qquad \Sigma(t_0) = \Sigma_0. \tag{38}$$

and

$$K(t) = \Sigma(t) H' W^{-1}. \tag{39}$$

The time-invariant infinite-horizon case coincides with the internal representation of the Wiener filter, which, like the Kolmogorov filter, was given in external form (Wiener, 1949).

3.3 The separation theorem

The control laws in Eq. (21), (25), or (29) are based on the exact measurement of the state $x(t)$. However, the underlying assumption of this theory is that only the output $y(t)$ is measured. The natural idea, then, is to associate a Kalman filter with the optimal linear-quadratic control law in an "optimum" observer-controller $u(t) = -C\hat{x}(t)$. This idea was proposed by

Kalman (1960a, c), who suggested that the duality principle legitimizes the association of the Kalman filter estimate with the optimal linear-quadratic gain. He was then anticipating the separation theorem. We now relate the results published in the year 1960 with the results that were to come later.

Optimality is a consequence of the *separation theorem*. This theorem was first proved by Joseph and Tou (1961) in discrete time, with no observation noise, as in Kalman (1960b), and, dually, no control cost, for a non-quadratic performance index. It was proved in continuous time by Wonham (1968). An early proof, more specific to the linear quadratic Gaussian case, was also due to Faurre (1968). "Certainty equivalence" results, in some particular cases with perfect state information and without formulation in terms of system theory, had appeared earlier in the economic literature (Simon, 1956; Theil, 1957).

The continuous-time problem is much more difficult, because part of the problem is to state the problem precisely. This involves continuous-time Brownian motions, Itô calculus, filtrations, and measurability. As Kalman in a conference on applications of the Kalman filter in hydrogeology, hydraulics, and water resources (1978) put it: "There are three types of filters: (i) those which keep tea leaves from falling into the tea cup, (ii) those we are talking about today, (iii) those which are so fancy that only topologists use them." We will not attempt to present this statement in modern terms, but rather stick to the form devised for engineers in the early sixties. Our purpose is to set firm grounds to devise a feed-back dynamic compensator (paragraph 4.1.3 below.)

Linear quadratic Gaussian (LQG) stochastic optimal control problem

Given a linear system in internal form with additive Gaussian white random disturbances in the dynamics and output equations, find a control law $u(.)$, if it exists, where $u(t)$ depends only on past outputs $y(\tau)$, for $\tau < t$, that minimizes the mathematical expectation of a quadratic performance index among all such control laws.

The answer is the *separation and certainty equivalence theorem*, holding true for discrete time and continuous time, finite-horizon problems, and stationary infinite-horizon problems:

Theorem 7 *The solution of the linear quadratic Gaussian stochastic optimal control problem exists and is obtained by replacing the state $x(t)$ by the Kalman filter estimate $\hat{x}(t)$ in the linear quadratic deterministic optimal control state feedback law. (Positive terms however must be added to Eq.* (22)

for the optimal criterion value.)

4 Applications

4.1 Compensator design in engineering

4.1.1 Linear and linearized control systems

Already in the “pre-Kalmanian” era, the linear time-invariant theory was used in various physical systems. Some had reasonable linear physical models, which were time-invariant when considered at the steady state. However, most industrial systems such as transportation or energy systems, demand a nonlinear model. The engineering practice, then, is to define a desired or *nominal* output trajectory, and take the *error signal* as the output of the control system, that is, the difference between the actual and the desired outputs. The objective of the control system is then to keep this error signal close to zero thanks to a *dynamic compensator*, which is a dynamic system whose input is the measured error signal, and the output the control input of the to-be-controlled system (that is, a feedback system, as understood by Wiener.)

In order to achieve this goal, one builds a linear model as the linearization of the nonlinear model for small deviations around the nominal trajectory. This can be done either on the basis of an analytic nonlinear model linearized by a first-order expansion, or of experiments using further parts of the theory (such as the consideration of cross-correlations between input and output pseudo-random small deviations).

Because the model is an approximation of the real physical system, the variables in the model are approximations of the physical variables. This is the "paradox of linear control theory;" the error stays close to zero because the control is efficient; the control is efficient because the system is well approximated by the model; the system is well approximated by the model because the linear approximation is good; the linear approximation is good because the error stays close to zero. The errors, which are approximated by the variables appearing in the control model, are then to be kept close to zero, in spite of disturbances in the dynamics, measurement errors, lack of direct measurement of some key variables, not to mention modeling errors and biases. These errors and biases differ from "noises" —here not a well-defined concept—, and were at the inception of "robust control" theories, and for our purpose, $\mathcal{H}_\infty$-optimal control by Zames (1981).

4.1.2 Observer-controller: algebraic procedure

Keeping a steady-state variable close to zero is achieved by forcing the system to be sufficiently stable. In that process, the poles of the transfer function, that is, the eigenvalues of the internal description of the overall system, are key information. The reason is that, for continuous-time systems, the real parts of these eigenvalues, which must be negative to insure stability, reflect the degree of stability. The imaginary parts measure the oscillatory behavior of the response of the system. (In discrete time, their modulus must be less than one to insure stability.)

As we mentioned in subsection 2.1.1, engineers had developed advanced tools beyond the mere inspection of the poles of the transfer function. These poles however remain of paramount importance, which justifies the use of the theorems on the separation of dynamics and on pole shifting, as well as the fact that the practitioner chooses the poles of the observer and the controller, such that these poles are independent of one another (this is the principle of separation of the dynamics). A rule of thumb is that the observer must be an order of magnitude faster than the controller. Trials and errors with the localization of the poles using simulation models (either linearized or nonlinear if available) allow one to construct an efficient control device.

4.1.3 Quadratic synthesis

Advancement of modeling science, which was largely due to digital computers, allowed the treatment of an increasingly high dimension of the models, to a point that simple methods based on the location of poles were no longer practical. For example, the rigid body dynamics of a landing airplane are described by a 12th-order system. Adding engines and control surface dynamics and flexible modes in modern airliners has pushed the order way beyond the 12th one.

Moreover, some problems, such as automatic landing of an airplane or control of an industrial baking cycle by heating and cooling an oven, intrinsically have their time horizon finite and are not linear time invariant, with emphasis sometimes on terminal error control. These problems are beyond the reach of algebraic methods.

Engineers may have a fairly precise idea of the origin and magnitude of the disturbances occurring in the dynamics and in the sources of error during measurements, let alone in biases inherent in the modeling. This provides a sound basis for using the Kalman filter to compute an observer's gain.

As for the controller, the control gain is computed through a quadratic performance index and the Kalman optimal gain. It is tuned by searching for satisfactory weighting matrices of the performance index by trials and errors. This operation is made easy and efficient if we understand the gain as the value minimizing this performance index. Then, one knows how the different state and control variables will respond to a change in the weighting matrices. This process, known as *quadratic synthesis*, is what made the new theory so popular among engineers, to the point of being the main tool used in designing the control systems for the Apollo mission.

4.2 Population ecology

The use of Kalman filtering in estimating a population, such as in a census, is nothing new. In the line of our presentation, we provide now an example of the use of linear quadratic optimal control in a rudimentary model of cooperation within an animal population. We show how the individual quest of self interest may sustain a collaborative behavior. We use this model to present several extensions of the linear quadratic theory, mainly:

- the problem comprises non-homogeneous terms,
- there is a time-discounted criterion,
- several players are involved, rather than a single optimizing one, and

there may exist a Nash equilibrium.

A large finite population of animals live in an area where individuals have two actions at their disposal: improve the foraging environment or forage. We have a colony of beavers constructing a dam in mind, for example. The food intake individuals obtain for a given foraging effort is an increasing function of environmental quality, and a decreasing one of total foraging effort, which depletes the resource. The quality of the environment decays exponentially when left unattended, but improves when animals invest in it, with a decreasing marginal effect. It may also be slightly degraded by the foraging effort.

We investigate the behavior at the Nash equilibrium among "selfish" individuals either over a season $[0, T]$ or over lifetime, with an exponentially distributed random time of death. Many models of evolutionary dynamics converge toward a Nash equilibrium, mainly so when, as here, this equilibrium is unique (Vincent and Brown, 2005; Sandholm, 2010: chap. 6).

We therefore introduce the scalar variables

- the quality x of the environment,
- the effort U_i produced by individual i to improve the (common) envi-

ronment, and $u_i = \sqrt{U_i}$ the decreasing marginal effect of effort,

- the foraging effort v_i of individual i,
- the efficiency factor K of any individual's foraging effort, in terms of energy intake,

and the (positive) parameters

- the population size n,
- the natural rate f of decay of the environment quality,
- the coefficient g of total effort $\sum_k u_k$ and h of total effort $\sum_k v_k$ in the dynamics governing environmental quality,
- the cost q of one unit of foraging effort and the cost r of one unit of effort devoted to improve the environment per time unit, in terms of energy consumption,
- the coefficients a and b in the affine function K (see below), and $\alpha = a - q$ assumed positive,
- the coefficient ρ of the exponential mortality rate.

The model is:

$$\dot{x} = -fx + g\sum_k u_k - h\sum_k v_k\,, \tag{40}$$

$$K = a + x - b\sum_k v_k\,, \tag{41}$$

Each individual (each beaver) seeks to maximize its performance index, which has the the same expression for the seasonal or for the lifetime optimization. In the first case, ρ may (optionally) be taken equal to zero (if the possibility of death during the season may be ignored); in the second case, $T = \infty$ and the stationary theory applies:

$$J_i = \int_0^T \mathrm{e}^{-\rho t}(Kv_i - qv_i - ru_i^2)\,\mathrm{d}t\,. \tag{42}$$

We analyze this problem in the appendix. We deal with the non homogeneous character by introducing a non-homogeneous Value function, with exponential discounting by introducing the same exponential term in the Value function, and with the many-player problem by looking for a Nash equilibrium, which we obtain by solving an optimization problem, where each individual assumes that all other players use the same optimal strategy.

The result is that cooperation is sustained. We introduce two coefficients P and p (variable in the finite-horizon case, fixed in the infinite-horizon case), and find that the equilibrium strategies are:

$$u^\star = \frac{g}{r}(Px + p)\,, \tag{43}$$

$$v^\star = \frac{1}{(n+1)b}\big((1 - 2hP)x - 2hp + \alpha\big) \tag{44}$$

For suitable values of the parameters, and in particular if n is large enough, Eq. (44) yields positive controls, with positive coefficients P and $(1 - 2hP)$, as well as stable dynamics and positive values for the quality x.

5 Carrying on

Kalman's contributions in the years 1960 and 1961 were a powerful stimulus for system theory and control research. Many researchers followed suit, both to get further theoretical advances (such as those mentioned in sections 3.3 and 4.2.2), and to develop algorithms implementing those theoretical results.

Algebraic system theory attracted many researchers such as Wonham (1967) and remained Kalman's main research area. It benefited from advanced algebraic tools and ideas coming from the theory of automata (Kalman, 1965; Kalman et al., 1969; Kalman, 1972)). The linear quadratic theory of control and observation was deeply renewed by the theory of $\mathcal{H}_\infty$-optimal control, pioneered by Zames (1981) in the classical "external" frequency-domain description, and later formulated in terms of time domain *à la* Kalman (Başar and Bernhard, 1995), with a minimax probability-free treatment of uncertainties, where the same duality shows up, in a more complex setup, and a bit mysteriously.

The basic theory quickly appeared in all textbooks of engineering control, and later in textbooks of many other domains (Weber, 2011). The algorithms were coded into publicly available software packages such as Matlab or Scilab, and they have been used in a wide range of application domains, well beyond the industrial and transportation systems of the early times, encompassing all branches of engineering as well as natural and bio-medical sciences.

Linear plus quadratic performance indices were used in the study of animal cooperation (Brown and Vincent, 2008). Yet, the *dynamic* linear-quadratic optimization theory is mostly absent from the literature on population dynamics. Most models used are nonlinear (Clark and Mangel, 2000; Pastor, 2008), such as the logistic growth model, but, as mentioned in paragraph 4.1.1, this is also true in other domains where the linear-quadratic optimization theory has had many applications. We have sketched here a possible use of the linear-quadratic optimal control theory in population ecology.

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A A model in population ecology

A.1 The Nash equilibrium

A.1.1 The Hamilton-Jacobi-Carathéodory-Isaacs-Bellman (HJCIB) equation

We assumed that all individuals are identical. Therefore they share a unique Bellman (or rather Isaacs Value) function, which we seek of the form $V(t,x) = \exp(-\rho t)W(t,x)$, with W a non-homogeneous quadratic function of x:

$$W(t,x) = P(t)x^2 + 2p(t)x + \pi(t)\,, \tag{45}$$

Thus, the HJCIB equation is

$$\begin{aligned} -\dot{W} + \rho W \;\; = \;\; \max_{u_i,v_i}\Big\{ & 2(Px+p)\left(-fx(t) + g\textstyle\sum_{k=1}^n u_k(t) - h\sum_{k=1}^n v_k\right) \\ & +v_i\left(\alpha + x(t) - b\textstyle\sum_{k=1}^n v_k(t)\right) - ru_i^2\Big\}. \end{aligned} \tag{46}$$

We derive the equations for the maximizing strategies $u^\star$ and $v^\star$ as

$$ru^\star = g(Px + p)\,, \tag{47}$$

$$(n+1)bv^\star = (1 - 2hP)x + \alpha - 2hp\,. \tag{48}$$

Putting these expressions back into the HJCIB Eq. (46) and equating the coefficients of the same powers of x, we obtain

$$\begin{aligned}
-\dot{P} &= \Big((2n-1)\tfrac{g^2}{r} + \tfrac{4n^2h^2}{(n+1)^2b}\Big)P^2 - 2\Big(f + \tfrac{\rho}{2} + \tfrac{(n^2+1)h}{(n+1)^2b}\Big)P + \tfrac{1}{(n+1)^2b}\,,\\
-\dot{p} &= \Big(-f - \rho + (2n-1)\tfrac{g^2}{r}P + \tfrac{4n^2h^2P-(n^2+1)h}{(n+1)^2b}\Big)p + \tfrac{1-(n^2+1)hP}{(n+1)2b}\alpha\,,\\
-\dot{\pi} &= -\rho\pi + \Big((2n-1)\tfrac{g^2}{r} - \tfrac{2h\alpha}{b}\Big) + \tfrac{(\alpha+2nhp)^2}{(n+1)^2b}\,.
\end{aligned} \tag{49}$$

These differential equations are to be integrated backward (numerically) from the terminal conditions (at T), which are equal to zero. The equation for P is decoupled from those for p and for π. Once P is computed, the equation for p is linear, and $\exp(-\rho t)\pi(t)$ is obtained as an integral.

A.1.2 Analysis in finite horizon

The equation for P is a Riccati equation $\dot{P} = -\Pi(P)$, where Π is a second-degree polynomial with two positive roots if these roots are real. The discriminant of Π is always positive if

$$f + \frac{\rho}{2} \geq \left(\frac{(2n-1)g^2}{(n+1)^2br}\right)^{\frac{1}{2}}, \tag{50}$$

(the right-hand side goes to zero as n increases) and positive otherwise if h is larger than a limit value that decreases as n increases. At terminal time T, P reaches 0 with a negative slope; as $t \to -\infty$, $P(t)$ converges toward the smallest (real) root $\bar{P}$ of Π if it exists. It diverges to $+\infty$ if not. (The possibility of P diverging, contrary to what is stated in Theorem 6, is due to a cross term xv in the criterion.) Therefore $P(t)$ remains positive, and smaller than $\bar{P}$ when it exists. $\bar{P}$ goes to zero as $1/n^2$ when n increases to infinity.

For h larger than a limit $h^\star$ (which goes to zero as n goes to ∞), the quantity

$$\Pi\left(\frac{1}{(n^2+1)h}\right) = \frac{(2n-1)g^2}{(n^2+1)^2 r} \times \frac{1}{h^2} - 2\frac{f+\frac{\rho}{2}}{n^2+1} \times \frac{1}{h} - \frac{1}{(n+1)^2 b} \qquad (51)$$

is negative, ensuring that a) Π has real roots, and b) $1-(n^2+1)hP$ and a fortiori $1-2hP$ are positive for all t. The equation for $-\dot{p}$ in Eq. (49) ensures that p is also positive for all t and smaller than a limit value $\bar{p}$, which goes to zero as $1/n$ when $n \to \infty$. Hence, for n large enough, if x is positive, so are $u^\star$ and $v^\star$.

Finally, the dynamics of environment quality under the equilibrium strategies are:

$$\dot{x} = \left(-f - \frac{nh}{(n+1)b} + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right) P\right) x + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right) p - \frac{nh\alpha}{(n+1)b}\,. \tag{52}$$

Therefore, x remains positive provided that α is smaller than a limit value, which is easy to find. (Actually, if $\alpha < 0$ but $x + \alpha > 0$, although $p < 0$, we still have $u^\star > 0$.)

A.1.3 Analysis in infinite horizon

In infinite horizon, if the algebraic Riccati Eq. (24) has real roots, then all coefficients P, p, and π are constant and equal to their limit values given by the algebraic equations obtained by setting the derivatives to zero. Closed form solutions are easy to write and useful for an efficient numerical implementation, but otherwise of little interest.

Existence of the positive limit value $\bar{p}$ requires that $1 - (n^2+1)h\bar{P} > 0$, which is a condition we have already examined, and

$$f + \rho + \frac{(n^2+1)h}{(n+1)^2 b} > \left((2n-1)\frac{g^2}{r} + \frac{4n^2h^2}{(n+1)^2 b}\right)\bar{P}\,, \tag{53}$$

again a condition which is satisfied, provided n is large.

The dynamics of environment quality are stable provided that

$$f + \frac{nh}{(n+1)b} > \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right)\bar{P}\,, \tag{54}$$

again a condition which is satisfied provided that n is large enough. It also ensures that $\mathrm{e}^{-\rho t}x^2$ is asymptotically stable, and that this stability is preserved if some of the players play $u_i = 0$. Both are necessary conditions for the Nash equilibrium in infinite horizon to exist (Mageirou, 1976).

A.2 Time domain versus frequency domain

With this elementary example, our main objective was to show

1. that Kalman's standard linear-quadratic theory can be extended in various directions, and

2. that some dynamic population ecology models may lead to such problems.

The infinite-horizon problem in a one-player version with $h = 0$ could be treated with the pre-Kalmanian theory of Newton Jr et al. (1957). In that case indeed, the v_i have no dynamic effect and may be chosen so as to maximize the integrand. Thus the problem becomes monovariable. The game version was more problematic, although Roos (1925) obtained the Cournot-Nash equilibrium of a linear quadratic differential game by the standard variational methods of the calculus of variations. The case with $h \neq 0$ was solvable in principle, provided that one performs a spectral factorization for the two-input multivariable system. The simplest way to achieve this task is with the Riccati Eq. (24), revived by Kalman (Willems, 1971). The finite-horizon seasonal problems were out of reach for this theory.